1. 已知:$|a|=\sqrt{13},|b|=\sqrt{19},|a+b|=\sqrt{24}$,则$|a-b|=$______

解： 由$a\cdot a=|a|^2$

$$(a+b)\cdot(a+b)=|a|^2+2a\cdot b+|b|^2$$

$$=13+19+2a\cdot b=24$$

$$a\cdot b=-4$$

$$(a-b)\cdot(a-b)=|a|^2-2a\cdot b+|b|^2$$

$$=13+19-2a\cdot b=32+8=40$$

$$|a-b|=\sqrt{40}$$

2.过点(2,0,–3)且与直线 $\begin{cases} x-2y+4z-7=0 \\ 3x+5y-2z+1=0 \end{cases}$

垂直的平面方程为________________________

解：$\vec{n}_1 = (1,-2,4), \vec{n}_2 = (3,5,-2)$

$$\vec{s} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & -2 & 4 \\ 3 & 5 & -2 \end{vmatrix} = -16i + 14j + 11k$$

令 $\vec{n} = \vec{s}$，平面：$-16(x-2) + 14y + 11(z+3) = 0$

3.求过原点及点$(6,-3,2)$且与平面$4x-y+2z=8$垂直的平面方程

解：过原点$Ax+By+Cz=0$,过点$(6,-3,2)$平面:$6A-3B+2C=0$

$$\vec{n}_1=(A,B,C);\vec{n}_2=(4,-1,2),$$

$$\vec{n}_1\cdot\vec{n}_2=0,\quad 4A-B+2C=0$$

$$\begin{cases}6A-3B+2C=0\\4A-B+2C=0\end{cases}\qquad A=B,C=-\frac{3}{2}B$$

平面为：$x+y-\dfrac{3}{2}z=0$

4.平面$3x-2y+6z-2=0$和平面$3x-2y+6z+12=0$之间的距离为:(　)

解：在平面$3x-2y+6z-2=0$上取一点$x=z=0, y=-1$

$(0,-1,0)$到$3x-2y+6z+12=0$的距离为d

$$d=\frac{|0+2+0+12|}{\sqrt{9+4+36}}=2$$

5.平面曲线:$\begin{cases}3x^2+2y^2=12\\z=0\end{cases}$ 绕y轴旋转一周而成的曲面在点$(0,\sqrt{3},\sqrt{2})$处的指向外侧的单位法向量为(　　　　)

解：旋转曲面为:$3x^2+3z^2+2y^2=12$

令$F(x,y,z)=3x^2+3z^2+2y^2-12,\vec{n}=(6x,4y,6z)$

$$\vec{n}\Big|_{M_0}=(0,4\sqrt{3},6\sqrt{2}),\vec{n}^0\Big|_{M_0}=\frac{1}{\sqrt{5}}(0,\sqrt{2},\sqrt{3})$$

6.求曲面$z-e^z+2xy=3$在点$(1,2,0)$处的切平面及法线方程。

解：令$F(x,y,z)=z-e^z+2xy-3$,

$$\vec{n}=(2y,2x,1-e^z)\qquad \vec{n}\Big|_{M_0}=(4,2,0)$$

切平面:$4(x-1)+2(y-2)=0$

法线:$\dfrac{x-1}{4}=\dfrac{y-2}{2}=\dfrac{z-0}{0}\Rightarrow\begin{cases}\dfrac{x-1}{2}=\dfrac{y-2}{1}\\ z=0\end{cases}$

7.求曲线$x=t, y=-t^2, z=t^3$与平面$x+2y+z=4$平行的切线

解:$\vec{T}=(1,-2t,3t^2), \vec{n}=(1,2,1)$

切线与平面平行，切向量与平面法向量垂直,$\vec{T}\cdot\vec{n}=0,$

$1-4t+3t^2=0, t_1=\dfrac{1}{3}, t_2=1$

$\vec{T}_1=(1,-\dfrac{2}{3},\dfrac{1}{3}), M_1(\dfrac{1}{3},-\dfrac{1}{9},\dfrac{1}{27}),$ 切线:$\dfrac{x-\frac{1}{3}}{1}=\dfrac{y+\frac{1}{9}}{-\frac{2}{3}}=\dfrac{z-\frac{1}{27}}{\frac{1}{3}}$

$\vec{T}_2=(1,-2,3), M_2(1,-1,1),$ 切线:$\dfrac{x-1}{1}=\dfrac{y+1}{-2}=\dfrac{z-1}{3}$

8.设 $\vec{AB}=(2,1,3),\vec{BC}=(-1,4,2)$,则$\Delta ABC$面积为 ______.

解：$s=\dfrac{1}{2}\left|\vec{AB}\times\vec{BC}\right|$,

$$\vec{AB}\times\vec{BC}=\begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ -1 & 4 & 2 \end{vmatrix}=-10i-7j+9k$$

$$\left|\vec{AB}\times\vec{BC}\right|=\sqrt{230}, \qquad s=\frac{1}{2}\sqrt{230}$$

9.求直线L:$\dfrac{x-1}{2}=\dfrac{y}{-1}=\dfrac{z-3}{1}$在平面$\pi$：$x-3y+2z-5=0$上投影直线。

解：直线可以等价于 $\begin{cases}\dfrac{x-1}{2}=\dfrac{y}{-1}\\ \dfrac{y}{-1}=\dfrac{z-3}{1}\end{cases} \Rightarrow \begin{cases}x-1+2y=0\\ y+z-3=0\end{cases}$

过直线的平面束方程：

$$x-1+2y+\lambda(y+z-3)=0$$

$$x+(2+\lambda)y+\lambda z-(3\lambda+1)=0$$

$$\vec{n}_1=(1,2+\lambda,\lambda),\vec{n}_2=(1,-3,2),$$

$$\vec{n}_1\perp\vec{n}_2,1-6-3\lambda+2\lambda=0$$

$\lambda=-5$， 垂直平面为：$x-3y-5z+14=0$

投影直线为：$\begin{cases}x-3y-5z+14=0\\ x-3y+2z-5=0\end{cases}$

方法2：平面法向量$\vec{n}=(1.-3,2)$，直线上面定点$A(1,0,3),B(3,-1,4)$

过A点作平面垂线为：$\dfrac{x-1}{1}=\dfrac{y-0}{-3}=\dfrac{z-3}{2}=\lambda, x=\lambda+1, y=-3\lambda, z=2\lambda+3,$

代入平面方程：

$$\lambda+1+9\lambda+4\lambda+6-5=0, 14\lambda=-2, \lambda=-\frac{1}{7}, \text{垂足}C(\frac{6}{7},\frac{3}{7},\frac{19}{7})$$

过B点作平面垂线为：$\dfrac{x-3}{1}=\dfrac{y+1}{-3}=\dfrac{z-4}{2}=t, x=t+3, y=-3t-1, z=2t+4$

代入平面方程：

$$t+3+9t+3+4t+8-5=0, 14t=-9, t=-\frac{9}{14}, \text{垂足}D(\frac{33}{14},\frac{13}{14},\frac{38}{14})$$

$$\vec{s}=\vec{CD}=(\frac{21}{14},\frac{7}{14},0)=\frac{1}{2}(3,1,0), CD\text{方程为}\frac{x-\frac{6}{7}}{3}=\frac{y-\frac{3}{7}}{1}=\frac{z-\frac{19}{7}}{0}$$

10.曲线$\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=2x \end{cases}$在$xoy$平面上的投影曲线为______________.

解：消去z得:$\begin{cases} x^2+y^2=2x \\ z=0 \end{cases}$

11.设$z=e^{\sin^2 xy}+x-1$,求dz

解：$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$

$$=(e^{\sin^2 xy}\cdot 2\sin xy\cdot\cos xy\cdot y+1)dx+e^{\sin^2 xy}\cdot 2\sin xy\cdot\cos xy\cdot xdy$$

$$=(ye^{\sin^2 xy}\sin 2xy+1)dx+xe^{\sin^2 xy}\sin 2xydy$$

12. $z=\frac{y}{x}f(xy)+y\cdot g(x^2+y^2)$, f和g有二阶连续偏导数，求$\frac{\partial^2 z}{\partial x\partial y}$

解：$$\frac{\partial z}{\partial x}=-\frac{y}{x^2}f(xy)+f'(xy)y\cdot\frac{y}{x}+g'(x^2+y^2)\cdot 2x\cdot y$$

$$=-\frac{y}{x^2}f(xy)+\frac{y^2}{x}f'(xy)+2xyg'(x^2+y^2)$$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{1}{x^2}f(xy)-f'(xy)\cdot x\cdot\frac{y}{x^2}+\frac{2y}{x}f'(xy)+f''(xy)\cdot x\cdot\frac{y^2}{x}+2xg'(x^2+y^2)+g''(x^2+y^2)\cdot 2y\cdot 2xy$$

$$=-\frac{1}{x^2}f(xy)-\frac{y}{x}f'(xy)+\frac{2y}{x}f'(xy)+y^2f''(xy)+2xg'(x^2+y^2)+4xy^2g''(x^2+y^2)$$

$$=-\frac{1}{x^2}f(xy)+\frac{y}{x}f'(xy)+y^2f''(xy)+2xg'(x^2+y^2)+4xy^2g''(x^2+y^2)$$

13.已知$e^{x+y+z}=x^2+y^2+z^2$,求$\frac{\partial y}{\partial z}$.

解：令$F(x,y,z)=e^{x+y+z}-x^2-y^2-z^2$

$$F_x=e^{x+y+z}-2x, F_y=e^{x+y+z}-2y, F_z=e^{x+y+z}-2z$$

$$\frac{\partial y}{\partial z}=-\frac{F_z}{F_y}=-\frac{e^{x+y+z}-2z}{e^{x+y+z}-2y}.$$

14.设$z=f(e^x\sin y,x^2+y^2)$, f具有二阶连续偏导数，求$\dfrac{\partial^2 z}{\partial x\partial y}$

解:$\dfrac{\partial z}{\partial x}=f_1\cdot e^x\sin y+f_2\cdot 2x=e^x\sin yf_1+2xf_2$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}(e^x\sin yf_1+2xf_2)=\frac{\partial}{\partial y}(e^x\sin yf_1)+\frac{\partial}{\partial y}(2xf_2)$$

$$=e^x\cos yf_1+e^x\sin y(f_{11}e^x\cos y+f_{12}\cdot 2y)+2x(f_{21}e^x\cos y+f_{22}2y)$$

$$=e^x\cos yf_1+e^{2x}\sin y\cos yf_{11}+2e^x(y\sin y+x\cos y)f_{12}+4xyf_{22}$$

15.求函数$u=x^2-2yz$在点$(1,-2,2)$处的方向导数最大值

解：方向导数最大值为梯度方向的方向导数, 大小为梯度的模

$$gradu=(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z})=(2x,-2z,-2y)$$

$$gradu\Big|_{(1,-2,2)}=(2,-4,4),\ \frac{\partial u}{\partial l}\Big|_{最大}=\Big|gradu\Big|_{(1,-2,2)}\Big|=6$$

16.求函数$f(x,y)=x^3-4x^2+2xy-y^2$的极值

解：$f_x=3x^2-8x+2y, f_y=2x-2y.$

令$f_x=3x^2-8x+2y=0$，$f_y=2x-2y=0$，

$x=y$,得$x=0, x=2$,从而得驻点$(0,0),(2,2)$

又$f_{xx}=6x-8, f_{xy}=2, f_{yy}=-2$。

(1)、驻点$(0,0)$处，$A=-8, B=2, C=-2.$

$AC-B^2=16-4>0, A<0$,有极大值$f(0,0)=0.$

(2)、驻点$(2,2)$处，$A=4, B=2, C=-2,$

$AC-B^2=-8-4<0$,无极值。

17.在平面$x+y+z=1$上求一点，使它与两定点$A(1,0,1),B(2,0,1)$的距离平方和最小。

解：设点为$M(x,y,z)$, $MA=\sqrt{(x-1)^2+y^2+(z-1)^2}$

$MB=\sqrt{(x-2)^2+y^2+(z-1)^2}$, $d=MA^2+MB^2=(x-1)^2+(x-2)^2+2y^2+2(z-1)^2$

设$L(x,y,z,\lambda)=(x-1)^2+(x-2)^2+2y^2+2(z-1)^2+\lambda(x+y+z-1)$

$$L_x=2(x-1)+2(x-2)+\lambda$$
$$L_y=4y+\lambda$$
$$L_z=4(z-1)+\lambda$$

令$\begin{cases} L_x=2(x-1)+2(x-2)+\lambda=0 \\ L_y=4y+\lambda=0 \\ L_z=4(z-1)+\lambda=0 \end{cases}$

显然:$4x-6=4y=4z-4$

将$x=y+\dfrac{3}{2}$, $z=y+1$代入, $x+y+z=1$得:

唯一驻点$(1,-\dfrac{1}{2},\dfrac{1}{2})$,即为所求。

18. $\iint\limits_D (x^2+xy-x)dxdy$, D由$y=x, y=2x$及$x=1$围成。

解:$\iint\limits_D (x^2+xy-x)dxdy=\int_0^1 dx\int_x^{2x}(x^2+xy-x)dy$

$$=\int_0^1 dx[x^2y+\frac{1}{2}xy^2-xy]_x^{2x}=\int_0^1(\frac{5}{2}x^3-x^2)dx=\frac{7}{24}$$

19. $\int_0^1 dx\int_{x^2}^1 \frac{xy}{\sqrt{1+y^3}}dy$

解：原式$=\int_0^1 dy\int_0^{\sqrt{y}}\frac{xy}{\sqrt{1+y^3}}dx=\int_0^1\frac{y}{\sqrt{1+y^3}}dy\int_0^{\sqrt{y}}xdx$

$$=\int_0^1\frac{y^2}{2\sqrt{1+y^3}}dy=\frac{1}{3}\int_0^1\frac{1}{2\sqrt{1+y^3}}dy^3$$

$$=\frac{1}{3}[\sqrt{1+y^3}]_0^1=\frac{1}{3}(\sqrt{2}-1)$$

20. $\iint\limits_D (x^2-2x+4xy)dxdy, D=\{(x,y): x^2+y^2\le 2x\}$

解：D上下对称，$4xy$为y的奇函数

$$\text{原式}=\iint\limits_D (x^2-2x)dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta}(r^2\cos^2\theta-2r\cos\theta)rdr$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\left[\frac{r^4}{4}\cos^2\theta-\frac{2}{3}r^3\cos\theta\right]_0^{2\cos\theta}=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(4\cos^6\theta-\frac{16}{3}\cos^4\theta)d\theta$$

$$=2\int_0^{\frac{\pi}{2}}(4\cos^6\theta-\frac{16}{3}\cos^4\theta)d\theta=2\int_0^{\frac{\pi}{2}}(4\sin^6\theta-\frac{16}{3}\sin^4\theta)d\theta$$

$$=8\int_0^{\frac{\pi}{2}}\sin^6\theta d\theta-\frac{32}{3}\int_0^{\frac{\pi}{2}}\sin^4\theta d\theta=8\times\frac{5}{6}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}-\frac{32}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$=-\frac{3}{4}\pi$$

21.$\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,dv$,$\Omega$为曲面$z=\sqrt{x^2+y^2}$和$z=\sqrt{1-x^2-y^2}$围成的立体。

解：交线为$x^2+y^2=\frac{1}{2}$，$z=\frac{\sqrt{2}}{2}, 0\le\theta\le 2\pi, 0\le\varphi\le\frac{\pi}{4}, 0\le r\le 1$

$$\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,dv=\int_0^{2\pi}d\theta\int_0^{\frac{\pi}{4}}d\varphi\int_0^1\sqrt{r^2\sin^2\varphi}\,r^2\sin\varphi\,dr$$

$$=2\pi\int_0^{\frac{\pi}{4}}d\varphi\int_0^1 r^3\sin^2\varphi\,dr=\frac{\pi}{2}\int_0^{\frac{\pi}{4}}\sin^2\varphi\,d\varphi=\frac{\pi}{4}\int_0^{\frac{\pi}{4}}(1-\cos 2\varphi)\,d\varphi$$

$$=\frac{\pi}{4}\left[\varphi-\frac{1}{2}\sin 2\varphi\right]_0^{\frac{\pi}{4}}=\frac{\pi^2}{16}-\frac{\pi}{8}$$

(特别注意：球面坐标下$x^2+y^2=r^2\sin^2\varphi$)

22. $\iiint\limits_{\Omega} z dx dy dz$, Ω为$x^2+y^2+z^2=4$与$x^2+y^2=3z$围成的立体。

解：交线为$x^2+y^2=3$，$z=1$

柱面坐标$0\le\theta\le 2\pi, 0\le r\le\sqrt{3}, \dfrac{r^2}{3}\le r\le\sqrt{4-r^2}$

$$\text{原式}=\int_0^{2\pi}d\theta\int_0^{\sqrt{3}}dr\int_{\frac{r^2}{3}}^{\sqrt{4-r^2}} z\cdot r dz=2\pi\int_0^{\sqrt{3}}dr[\frac{1}{2}rz^2]_{\frac{r^2}{3}}^{\sqrt{4-r^2}}$$

$$=\pi\int_0^{\sqrt{3}}(4r-r^3-\frac{r^5}{9})dr$$

$$=\pi[2r^2-\frac{r^4}{4}-\frac{r^6}{54}]_0^{\sqrt{3}}=\frac{13}{4}\pi$$

(注意球面坐标不方便取r的区间)

(23).已知$z=u^2\sin v, u=\dfrac{x}{y}, v=xe^y$,求$\dfrac{\partial z}{\partial x}$

解:$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}$

$$=2u\sin v\cdot\frac{1}{y}+u^2\cos v\cdot e^y$$

$$=\frac{2x}{y^2}\sin(xe^y)+\frac{x^2}{y^2}e^y\cos(xe^y)$$

(24).求$u=xy^2+z^3-xyz$在点(1,1,2)沿方向角为:

$\alpha=\frac{\pi}{3},\beta=\frac{\pi}{4},\gamma=\frac{\pi}{3}$的方向的方向导数。

解:$\frac{\partial u}{\partial x}=y^2-yz,\frac{\partial u}{\partial y}=2xy-xz,\frac{\partial u}{\partial z}=3z^2-xy$

$$\left.\frac{\partial u}{\partial x}\right|_{(1,1,2)}=-1,\left.\frac{\partial u}{\partial y}\right|_{(1,1,2)}=0,\left.\frac{\partial u}{\partial z}\right|_{(1,1,2)}=11,$$

$$e=(\cos\alpha,\cos\beta,\cos\gamma)=(\frac{1}{2},\frac{\sqrt{2}}{2},\frac{1}{2})$$

$$\left.\frac{\partial u}{\partial l}\right|_{(1,1,2}=(-1,0,11)\cdot(\frac{1}{2},\frac{\sqrt{2}}{2},\frac{1}{2})=5$$

(25).求底圆半径相等的两个直交圆柱面$x^2+y^2=R^2, x^2+z^2=R^2$所围立体的表面积。

解：所围立体在8个卦限都对称，求第一卦限的表面积s_1

第一卦限表面积为2部分，一部分为$x^2+y^2=R^2$的，

另一部分为$x^2+z^2=R^2$的，面积相等。

求$x^2+z^2=R^2$在第一卦限表面积，$z=\sqrt{R^2-x^2}$,投影区域为：

$$0\le\theta\le\frac{\pi}{2}, 0\le r\le R(0\le x\le R, 0\le y\le\sqrt{R^2-x^2})$$

$$s=16\iint_{D_1}\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dxdy=16\iint_{D_1}\sqrt{1+(\frac{-x}{\sqrt{R^2-x^2}})^2}dxdy$$

$$=16R\iint_{D_1}\frac{1}{\sqrt{R^2-x^2}}dxdy=16R\int_0^R dx\int_0^{\sqrt{R^2-x^2}}\frac{1}{\sqrt{R^2-x^2}}dy$$

$$=16R^2$$

23.求球面$x^2+y^2+z^2=a^2$含在圆柱面$x^2+y^2=ax$内部的那部分面积和球体$x^2+y^2+z^2\le a^2$含在圆柱面$x^2+y^2=ax$内部的那部分体积。

(1)由对称性，第一卦限为$s_1=\iint\limits_D\sqrt{1+z_x{}^2+z_y{}^2}\,dxdy$

$$z=\sqrt{a^2-x^2-y^2},z_x=\frac{-x}{\sqrt{a^2-x^2-y^2}},z_y=\frac{-y}{\sqrt{a^2-x^2-y^2}}$$

$$s=4s_1=4\iint\limits_D\sqrt{1+\frac{x^2}{a^2-x^2-y^2}+\frac{y^2}{a^2-x^2-y^2}}dxdy$$

$$=4\iint\limits_D\frac{adxdy}{\sqrt{a^2-x^2-y^2}}=4a\int_0^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\frac{r}{\sqrt{a^2-r^2}}dr=4a\int_0^{\frac{\pi}{2}}d\theta[-\sqrt{a^2-r^2}]_0^{a\cos\theta}$$

$$=4a\int_0^{\frac{\pi}{2}}(a-a\sin\theta)d\theta=4a^2(\frac{\pi}{2}-1)=2\pi a^2-4a^2$$

23.求球面$x^2+y^2+z^2=a^2$含在圆柱面$x^2+y^2=ax$内部的那部分面积和球体$x^2+y^2+z^2\le a^2$含在圆柱面$x^2+y^2=ax$内部的那部分体积。

$$(2) z=\sqrt{a^2-x^2-y^2},$$

$$v=4v_1=4\iint_D z\,dxdy=4\iint_D \sqrt{a^2-x^2-y^2}\,dxdy$$

$$=4\int_0^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}\sqrt{a^2-r^2}\cdot r\,dr$$

$$=2\int_0^{\frac{\pi}{2}}d\theta\int_0^{a\cos\theta}-\sqrt{a^2-r^2}\,d(a^2-r^2)$$

$$=2\int_0^{\frac{\pi}{2}}d\theta[-\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}]_0^{a\cos\theta}=\frac{4}{3}\int_0^{\frac{\pi}{2}}(a^3-a^3\sin^3\theta)d\theta$$

$$=\frac{4a^3}{3}(\frac{\pi}{2}-\int_0^{\frac{\pi}{2}}\sin^3\theta d\theta)=\frac{4a^3}{3}(\frac{\pi}{2}-\frac{2}{3})$$

24.求锥面$z=\sqrt{x^2+y^2}$被柱面$z^2=2x$所割下的那部分面积。

解：曲面为：$z=\sqrt{x^2+y^2}$，$z_x=\dfrac{x}{\sqrt{x^2+y^2}}, z_y=\dfrac{y}{\sqrt{x^2+y^2}}$.

$z=\sqrt{x^2+y^2}$与$z^2=2x$交线为:$(x-1)^2+y^2=1$.

D_{xy}为:$(x-1)^2+y^2\le 1$

$$s=\iint\limits_{D_{xy}}\sqrt{1+z^2{}_x+z^2{}_y}\,dxdy=\iint\limits_{D_{xy}}\sqrt{1+1}\,dxdy=\sqrt{2}\pi$$

25.一物体占有的闭区域由曲面$z=x^2+y^2$和平面$z=0,|x|=a,|y|=a$所围成，求其体积。

$$v=\iint_D f(x,y)dxdy=\int_{-a}^{a}dx\int_{-a}^{a}(x^2+y^2)dy$$

$$=4\int_0^a dx\int_0^a(x^2+y^2)dy=4\int_0^a dx(ax^2+\frac{1}{3}a^3)$$

$$=4\int_0^a (ax^2+\frac{1}{3}a^3)dx=4[\frac{a}{3}x^3]_0^a+4a\times\frac{1}{3}a^3=\frac{8}{3}a^4$$

26.设曲线$L: x^2+y^2=R^2$,求$\int_L (x+2y)^2 ds$

解:$\int_L (x+2y)^2 ds=\int_L (x^2+4xy+4y^2)ds$

$$=\int_L (R^2+4xy+3y^2)ds$$

$$=\int_0^{2\pi} (R^2+4R^2\cos t\sin t+3R^2\sin^2 t)\sqrt{x_t'^2+y_t'^2}dt$$

$$=R^3\int_0^{2\pi} (1+4\cos t\sin t+3\sin^2 t)dt$$

$$=R^3\int_0^{2\pi} (1+4\cos t\sin t+3\frac{1-\cos 2t}{2})dt=5\pi R^3$$

27.设曲线$L: y=\sqrt{1-x^2}\ \ (-1\le x\le 1)$,求$\int_L (x^2+2xy)ds$

解:$\int_L (x^2+2xy)ds=\int_{-1}^{1}(x^2+2x\sqrt{1-x^2})\sqrt{1+y'^2}dx$

$$=\int_{-1}^{1}x^2\sqrt{1+y'^2}dx=2\int_0^1 x^2\sqrt{1+(\frac{-x}{\sqrt{1-x^2}})^2}dx$$

$$=2\int_0^1\frac{x^2}{\sqrt{1-x^2}}dx=2\int_0^1(-\sqrt{1-x^2}+\frac{1}{\sqrt{1-x^2}})dx$$

$$=-\frac{\pi}{2}+2\times\frac{\pi}{2}=\frac{\pi}{2}$$

28. $\int_L x^2 y dx + xy^2 dy$, 其中 $L: |x| + |y| = 1$, 逆时针方向。

解: $\int_L x^2 y dx + xy^2 dy = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \iint_D (y^2 - x^2) dxdy$

$$= \int_{-1}^{0} dx \int_{-x-1}^{1+x} (y^2 - x^2) dy + \int_{0}^{1} dx \int_{x-1}^{1-x} (y^2 - x^2) dy = 0$$

29. $\int_{(1,1)}^{(2,2)} xy^2 dx + x^2 y dy$,

$P = xy^2, Q = x^2 y, \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} = 2xy$, 曲线积分与路径无关。

$$\int_{(1,1)}^{(2,2)} xy^2 dx + x^2 y dy = \int_{(1,1)}^{(2,1)} xy^2 dx + x^2 y dy + \int_{(2,1)}^{(2,2)} xy^2 dx + x^2 y dy$$

$$= \int_{(1,1)}^{(2,1)} xy^2 dx + \int_{(2,1)}^{(2,2)} x^2 y dy = \int_1^2 x dx + \int_1^2 4y dy = 7\frac{1}{2}.$$

30. $\int_L (e^x \sin y + x - y)dx + (e^x \cos y + y)dy$, L是圆周$y = \sqrt{2ax - x^2}$从点$A(2a,0)$到$O(0,0)$的弧段。

解：$\int_{L+OA} (e^x \sin y + x - y)dx + (e^x \cos y + y)dy$

$$= \iint_D (e^x \cos y - e^x \cos y + 1)dxdy = \frac{\pi a^2}{2}.$$

$$\int_L (e^x \sin y + x - y)dx + (e^x \cos y + y)dy$$

$$= \frac{\pi a^2}{2} - \int_{OA} (e^x \sin y + x - y)dx + (e^x \cos y + y)dy$$

$$= \frac{\pi a^2}{2} - \int_{OA} xdx = \frac{\pi a^2}{2} - \int_0^{2a} xdx = \frac{\pi a^2}{2} - 2a^2$$

31.计算$\int_L (e^x+1)\cos y dx-(e^x+x)\sin y dy$，其中$L$是由点$A(2,0)$沿心形线$r=1+\cos\theta$上侧到原点$O$的弧。

$$解:\int_{L+OA} Pdx+Qdy=\iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$$

$$=\iint_D [-(e^x+1)\sin y+(e^x+1)\sin y]dxdy=0$$

$$\int_L (e^x+1)\cos y dx-(e^x+x)\sin y dy$$

$$=0-\int_{OA} (e^x+1)\cos y dx-(e^x+x)\sin y dy$$

$$=-\int_{OA} (e^x+1)dx=-\int_0^2 (e^x+1)dx=-e^2-1.$$

32.已知$(2x\cos y+y^2\cos x)dx+(2y\sin x-x^2\sin y)dy$为$u(x,y)$的全微分，求$u(x,y)$

解：$P=2x\cos y+y^2\cos x, Q=2y\sin x-x^2\sin y$

$$\frac{\partial Q}{\partial x}=2y\cos x-2x\sin y, \frac{\partial P}{\partial y}=-2x\sin y+2y\cos x$$

$$\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}, \Rightarrow du=(2x\cos y+y^2\cos x)dx+(2y\sin x-x^2\sin y)dy,$$

$$u(x,y)=\int_{(x_0,y_0)}^{(x,y)}(2x\cos y+y^2\cos x)dx+(2y\sin x-x^2\sin y)dy,$$

取$(x_0,y_0)=(0,0)$

$$u(x,y)=\int_{(0,0)}^{(x,0)}(2x\cos y+y^2\cos x)dx+(2y\sin x-x^2\sin y)dy$$

$$+\int_{(x,0)}^{(x,y)}(2x\cos y+y^2\cos x)dx+(2y\sin x-x^2\sin y)dy$$

$$=\int_0^x 2xdx+\int_0^y(2y\sin x-x^2\sin y)dy$$

$$=x^2+[y^2\sin x+x^2\cos y]_0^y=x^2+y^2\sin x+x^2\cos y-x^2$$

$$=y^2\sin x+x^2\cos y$$

33. $\iint\limits_{\Sigma} z^2 ds$, Σ是锥面$z=\sqrt{x^2+y^2}$位于$z=2$下方的部分。

解：$z_x=\dfrac{x}{\sqrt{x^2+y^2}}, z_y=\dfrac{y}{\sqrt{x^2+y^2}}, ds=\sqrt{1+z_x{}^2+z_y{}^2}dxdy$

$$D_{xy}: x^2+y^2\le 4$$

$$\iint\limits_{\Sigma} z^2 ds=\iint\limits_{D}(x^2+y^2)\sqrt{1+z_x{}^2+z_y{}^2}dxdy=\sqrt{2}\iint\limits_{D}(x^2+y^2)dxdy$$

$$=\sqrt{2}\int_0^{2\pi}d\theta\int_0^2 r^3dr=8\sqrt{2}\pi$$

34.求$I=\iint\limits_{\Sigma} z\,ds$,其中$\Sigma$为$x^2+y^2+z^2=1, z=\sqrt{x^2+y^2}$所围成的立体表面。

解:$\Sigma_1: z=\sqrt{1-x^2-y^2}$, $\Sigma_2: z=\sqrt{x^2+y^2}$,交线: $x^2+y^2=\frac{1}{2}$, $D_{xy}: x^2+y^2\le\frac{1}{2}$.

$$\Sigma_1, z_x=\frac{-x}{\sqrt{1-x^2-y^2}}, z_y=\frac{-y}{\sqrt{1-x^2-y^2}}, ds=\sqrt{1+z_x^{\ 2}+z_y^{\ 2}}dxdy=\frac{dxdy}{\sqrt{1-x^2-y^2}}$$

$$\Sigma_2, z_x=\frac{x}{\sqrt{x^2+y^2}}, z_y=\frac{y}{\sqrt{x^2+y^2}}, ds=\sqrt{1+z_x^{\ 2}+z_y^{\ 2}}dxdy=\sqrt{2}dxdy$$

$$I=\iint\limits_{\Sigma} zds=\iint\limits_{\Sigma_1} zds+\iint\limits_{\Sigma_2} zds=\iint\limits_{D_{xy}}\sqrt{1-x^2-y^2}\frac{dxdy}{\sqrt{1-x^2-y^2}}+\sqrt{2}\iint\limits_{D_{xy}}\sqrt{x^2+y^2}dxdy$$

$$=\frac{1}{2}\pi+\sqrt{2}\int_0^{2\pi}d\theta\int_0^{\frac{1}{\sqrt{2}}}r^2dr=\frac{1}{2}\pi+2\sqrt{2}\pi\frac{1}{3}\frac{1}{2\sqrt{2}}=\frac{5\pi}{6}$$